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BENDING WAVES IN IMPULSIVELY LOADED ELASTIC–PLASTIC PLATES

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Abstract—The paper considers bending waves in bilinear elastic–plastic plates prestressed beyond the yield point. The strain reversals associated with the bending motion are found to give the bend a velocity that depends only on the material properties of the plate, and lies between the longitudinal elastic and plastic wave velocities.

The section of maximum curvature is found to have a higher velocity than other sections in the bend. The wave will therefore develop a curvature discontinuity at the front, where the curvature jumps from zero to its maximum. The slower running tail of the wave is left behind, and the bend therefore widens.

INTRODUCTION

WHEN a supported plate is impulsively loaded, both longitudinal and transverse waves radiate from the supports. According to Taylor [1] the angle of plate deflection θ due to the transverse wave is given by

$$\operatorname{tg}\theta = \frac{V}{U},$$

where V is the normal velocity of the plate inflicted by the load, and U is the velocity of the transverse wave (Fig. 1). While V is a function of the load and the plate mass, U will depend on the material properties of the plate.

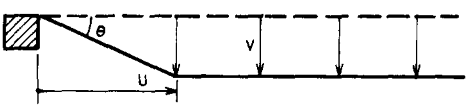

FIG. 1. Simple model of plate deformation. Information about the support is brought along the plate with velocity U. The plate motion with velocity V is stopped, and the plate bent at an angle $\theta = \operatorname{arctg} V/U$.

The approximate theory for transverse waves in bars due to Timoshenko [2, 3], and extended to plates by Mindlin [4], gives a good description of the elastic waves. No similar theory for plastic waves, or rather transverse waves in bars and plates in the plastic range, seems to exist. Attempts have been made to solve the problem of dynamic plastic bending by extending the elastic flexural solution [5], by using rigid–perfectly plastic analysis [6], and by superposition of elastic and plastic harmonics [7, 8]. The propagation of plastic bending waves in a bar of strain rate material has been studied by Plass [9]. The strain rate dependence is essential to his results, since the signal velocities in the bar then are equal to the elastic characteristic velocities. The solution is therefore not valid for materials that do not exhibit strain rate effects.

In the impulsively loaded plate the transverse wave is superimposed on the elastic–plastic wave system. In a bilinear elastic–plastic material this longitudinal wave system raises the stress in two steps—the elastic wave raises it to the yield point, and later the plastic wave raises it to the level appropriate to the load at the end. The velocity of the transverse wave cannot be greater than the velocity of the longitudinal elastic wave. Except for the cases when the velocity of the transverse wave is equal to either of the longitudinal wave velocities, the plate material is therefore in the plastic range as the bend approaches, with the same longitudinal stress in front of and behind the bend.

This paper therefore describes an attempt to find the velocity of the bend U in a bilinear elastic–plastic plate prestressed into the plastic range. The neutral plane is inside the plate during the bending, and both loading and unloading occur. Part of each section is therefore in the elastic range, part is in the plastic range. Characteristic velocities in the bend lie between the elastic and the plastic velocities. The analysis indicates that these velocities increase with the thickness of the elastic part of the section, making the section of least plastic part the fastest section in the bend. This section is the section of maximum curvature, and consequently the section of maximum curvature is at the front of the bend.

FORMULATION OF THE PROBLEM

It is assumed that a bending wave, travelling in the x-direction, has been generated in a prestressed bilinear elastic–plastic plate of thickness $2h$. The prestress is also in the x-direction, and the stress–strain curve for the material is as shown in Fig. 2. The slope of

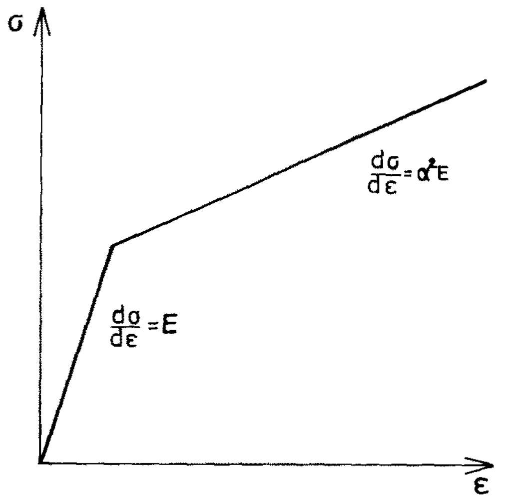

FIG. 2. Stress–strain curve for the plate material under uniaxial stress.

the curve, $d\sigma/d\varepsilon$, is equal to Youngs modulus E under elastic deformations, and to $\alpha^2 E$ under plastic deformations. α^2 is then the coefficient of workhardening.

With sufficiently high preload the plate is completely plastic when the bending wave arrives. The load is supposed constant during the bending motion, and the longitudinal stress changes associated with the bend must therefore cancel over each section. Thus the increased load in one part of the section must be balanced by unloading in another part.

In the most general case a single crested bending wave may be divided into two regions— one of increasing curvature, region I, and one of decreasing curvature, region II, as indicated in Fig. 3. As the curvature increases the outside of the bend is stretched and loaded, while

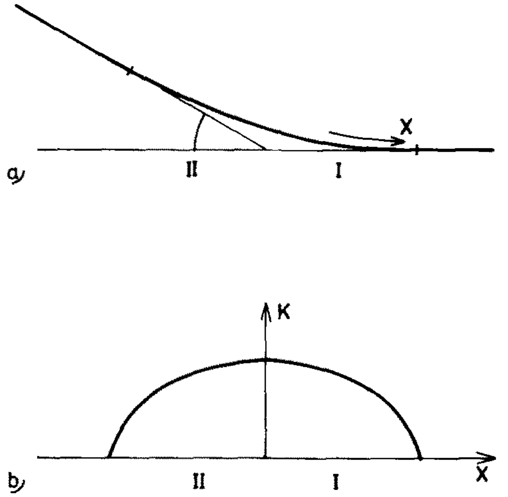

FIG. 3. A bend (a) and its corresponding curvature wave (b). The curvature K increases in region I, and decreases in region II.

the inside is compressed and unloaded (Fig. 4a). The deformation starts from a uniform strain level, and is easily described, with plastic loading and elastic unloading. As the curvature is reduced in region II strain reversals must occur at least in parts of the section (Fig. 5a). These reversals start from a nonuniform strain distribution, and while the unloading is elastic, the loading will be both elastic and plastic.

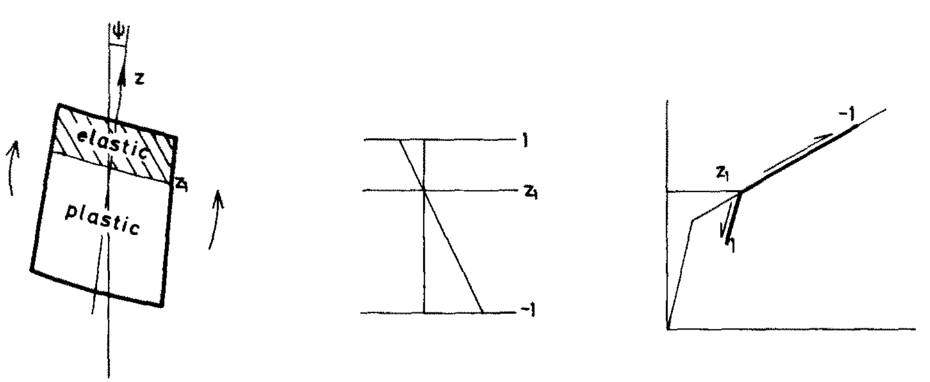

FIG. 4. A segment of the plate in region I, under increasing curvature, (a), and the corresponding strain distribution (b) and stress changes (c). The stress σ_0 corresponds to the neutral plane at $z = z_1$.

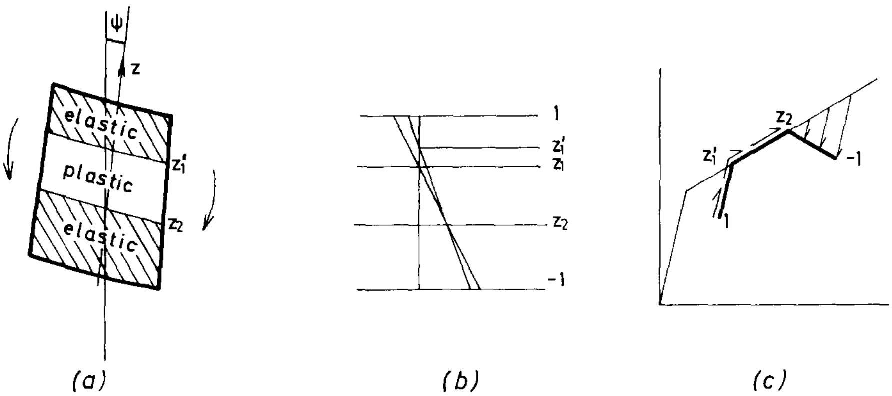

FIG. 5. A segment of the plate in region II, under decreasing curvature, (a), and the corresponding strain distribution (b), and stress changes (c).

STRESS–STRAIN RELATIONS AND EQUATIONS OF MOTION

The surfaces of the plate are free of stress. If the stress along the plate normal is assumed to be zero all through the plate, the stress–strain relations in the plate corresponding to the stress–strain curve of Fig. 2 are for purely elastic deformations,

$$\begin{aligned} E\varepsilon_x &= \sigma_x - \nu_1\sigma_y \\ 0 &= -\nu_1\sigma_x + \sigma_y \\ E\varepsilon_z &= -\nu_1(\sigma_x + \sigma_y), \end{aligned} \qquad (1)$$

and for plastic deformations,

$$\begin{aligned} \alpha^2 E\varepsilon_x &= \sigma_x - \nu_2\sigma_y \\ 0 &= -\nu_2\sigma_x + \sigma_y \\ \alpha^2 E\varepsilon_z &= -\nu_2(\sigma_x + \sigma_y), \end{aligned} \qquad (2)$$

where ν_1 is Poissons ratio, and ν_2 is the analogous plastic Poissons ratio [10]. In a perfectly plastic material plastic deformations are without a recoverable elastic component, $\nu_2 = \frac{1}{2}$, while for perfectly recoverable deformations $\nu_2 = \nu_1$, i.e. $\nu_1 < \nu_2 < \frac{1}{2}$.

From (1) and (2) the longitudinal stress–strain relation in the plate can be expressed by

$$E\varepsilon_x = (1-\nu_1^2)\sigma_x$$

and

$$\alpha^2 E\varepsilon_x = (1-\nu_2^2)\sigma_x$$

for elastic and plastic deformations respectively.

When dimensionless stress is introduced by

$$\sigma = \frac{1}{\rho_0 c_p^2}\sigma_x = \frac{1-\nu_1^2}{E}\sigma_x$$

where

$$c_p = \left\{\frac{E}{\rho_0(1-\nu_1^2)}\right\}^{\frac{1}{2}}$$

is the elastic plate velocity and ρ_0 is the density of the undisturbed plate, the stress–strain relations take the form

$$\sigma = \varepsilon_x = \varepsilon$$

for elastic deformations, and

$$\sigma = \frac{1-\nu_1^2}{1-\nu_2^2}\alpha^2\varepsilon = \beta^2\varepsilon \tag{3}$$

for plastic deformations, where β then is the ratio between the elastic and plastic plate velocities.

When x is a Lagrangian coordinate in the direction of prestress and wave propagation, and z is Lagrangian coordinate originally along the plate normal, the longitudinal equation of motion for the plate is

$$\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{xz}}{\partial z} = \rho_0\frac{\partial^2 u}{\partial t^2} \tag{4}$$

where τ_{xz} is the shear stress on planes with normals in the x- and z-directions, and u is the displacement in the x-direction.

In terms of the dimensionless variables

$$x' = \frac{x}{h}, \qquad z' = \frac{z}{h}, \qquad u' = \frac{u}{h}, \qquad t' = \frac{c_p t}{h},$$

h being half the plate thickness, the equation of motion (4) is

$$\frac{\partial \sigma}{\partial x'}+\frac{\partial \tau}{\partial z'} = \frac{\partial^2 u}{\partial t'^2}, \tag{5}$$

where

$$\tau = \frac{1}{\rho_0 c_p^2}\tau_{xz}.$$

To find the velocity of the bending wave, equation (5) is needed in its integrated form

$$\int_{-1}^{1}\frac{\partial \sigma}{\partial x}z\,\mathrm{d}z+\int_{-1}^{1}\frac{\partial \tau}{\partial z}z\,\mathrm{d}z = \int_{-1}^{1}\frac{\partial^2 u}{\partial t^2}z\,\mathrm{d}z, \tag{6}$$

or

$$\frac{\partial M}{\partial x}-Q = \frac{\partial^2}{\partial t^2}\int_{-1}^{1} uz\,\mathrm{d}z, \tag{7}$$

where

$$M = \int_{-1}^{1}\sigma z\,\mathrm{d}z \quad \text{and} \quad Q = \int_{-1}^{1}\tau\,\mathrm{d}z$$

are the bending moment and the shear force on the section, and the integration is over the dimensionless plate thickness from -1 to 1.

DETERMINATION OF ELASTIC AND PLASTIC ZONES IN THE SECTION

The condition that the longitudinal force in the plate is constant is expressed by

$$\int_{-1}^{1} \sigma \, dz = 2\sigma_0, \tag{8}$$

where σ_0 is the prestress.

In the region I of increasing curvature the stress may be written $\sigma = \sigma_0 + \sigma_1$, where σ_1 is the deviation from the average, expressed by $\sigma_1 = \varepsilon$ for unloading and $\sigma_1 = \beta^2\varepsilon$ for loading from the completely plastic state at the stress level σ_0 (Fig. 4c).

If plane sections remain plane under the deformation,* and the angle a section is turned is ψ (Fig. 4a), the strain in region I can be expressed by

$$\varepsilon = -(z - z_1)\frac{\partial \psi}{\partial x},$$

where z_1 is the coordinate of the neutral plane. Strains are considered positive in extension, and the curvature $\partial\psi/\partial x$ is positive. The stress deviation σ_1 is then expressed by

$$\sigma_1 = -(z - z_1)\frac{\partial \psi}{\partial x}$$

under unloading, and

$$\sigma_1 = -\beta^2(z - z_1)\frac{\partial \psi}{\partial x}$$

under loading. The stress integral (8) then gives

$$\int_{-1}^{1} \sigma \, dz = \int_{-1}^{1} \sigma_0 \, dz - \frac{\partial \psi}{\partial x}\left\{\beta^2 \int_{-1}^{z_1} (z - z_1) \, dz + \int_{z_1}^{1} (z - z_1) \, dz\right\} = 2\sigma_0,$$

and hence

$$\tfrac{1}{2}(1-\beta^2)z_1^2 - (1+\beta^2)z_1 + \tfrac{1}{2}(1-\beta^2) = \tfrac{1}{2}(1-\beta^2)\left(z_1 - \frac{1-\beta}{1+\beta}\right)\left(z_1 - \frac{1+\beta}{1-\beta}\right) = 0.$$

Since the neutral plane must be inside the section, $|z_1| \leq 1$, this gives

$$z_1 = \frac{1-\beta}{1+\beta}.$$

The position of the neutral plane is constant in the region I of increasing curvature, giving a strain distribution as shown in Fig. 4b.

* This generally implies that elastic and plastic deformations in bending are of the same order of magnitude.

In the region where the curvature decreases, the stress can be expressed by $\sigma = \sigma_0 + \sigma_1 + \sigma_2$, where again σ_0 is the prestress, σ_1 is the final stress in region I, and σ_2 is the additional stress in region II. The stress integral (8) now takes the form

$$\int_{-1}^{1} \sigma_0 \, dz + \int_{-1}^{1} \sigma_1 \, dz + \int_{-1}^{1} \sigma_2 \, dz = 2\sigma_0,$$

and since

$$\int_{-1}^{1} \sigma_0 \, dz = 2\sigma_0 \quad \text{and} \quad \int_{-1}^{1} \sigma_1 \, dz = 0$$

as shown above, this leads to

$$\int_{-1}^{1} \sigma_2 \, dz = 0 \tag{9}$$

at any section in region II.

The decrease in curvature must give strain reversals relative to the final strain distribution of region I. If $z = z_2$ is the position of the neutral plane in region II, lamina at $z < z_2$ will experience elastic unloading (Fig. 5a). Since the stress changes must cancel, this implies that $z_2 < z_1$, resulting in a strain distribution as indicated in Fig. 5b. The lamina in the neighbourhood of the old neutral plane will then all be stretched. On one side there will be further plastic loading, while the previously unloaded lamina will experience strain reversals. This will give elastic loading until the yield point is reached at $\sigma = \sigma_0$, and from then on the loading will be plastic. The thickness of the first elastic zone is thereby reduced, giving a change in the distribution of stress increments that must be accounted for by a change in the position of the neutral plane. The elastic–plastic boundaries at $z = z_1'$ and $z = z_2$ will therefore move until the first elastic zone disappears as $z_1' = 1$. The stress changes in region II from the final distribution of region I are indicated in Fig. 5c.

With continuous changes in z_1' and z_2 the additional strains of region II will not be expressed by a simple function of z, and it is difficult to determine the position of the neutral plane from the stress condition (9). However, the initial strain distribution of region II must be the same as the final distribution of region I. In the limit the initial strain changes in region II can then be expressed by

$$\Delta\varepsilon = -(z - z_2)\Delta\left(\frac{\partial\psi}{\partial x}\right)$$

and the corresponding stress changes by

$$\Delta\sigma = \Delta\varepsilon = -(z - z_2)\Delta\left(\frac{\partial\psi}{\partial x}\right)$$

and

$$\Delta\sigma = \beta^2\Delta\varepsilon = -\beta^2(z - z_2)\Delta\left(\frac{\partial\psi}{\partial x}\right)$$

for elastic and plastic deformations respectively. The initial position of the neutral plane is then found from the stress condition (9)

$$\Delta \int_{-1}^{1} \sigma \, dz = \int_{-1}^{1} \Delta\sigma \, dz$$

$$= -\Delta\left(\frac{\partial \psi}{\partial x}\right)\left\{\int_{-1}^{z_2} (z-z_2)\, dz + \beta^2 \int_{z_2}^{z_1'} (z-z_2)\, dz + \int_{z_1'}^{1} (z-z_2)\, dz\right\} = 0, \quad (10)$$

with $z_1' = z_1$. Hence

$$\tfrac{1}{2}(1-\beta^2)(z_1-z_2)^2 + 2z_2 = 0$$

or

$$z_2 = -\frac{\{1-\sqrt{[1-(1-\beta^2)z_1]}\}^2}{1-\beta^2} = -\frac{[1-\sqrt{(2\beta-\beta^2)}]^2}{1-\beta^2} \quad (11)$$

where the sign of the root has been chosen to make $z_2 > -1$, to keep the neutral plane inside the section.

As the curvature decreases the width of the first elastic zone is reduced, and it finally disappears as $z_1' = 1$. From then on the conditions are similar to those in region I, with plastic loading on one side of the neutral plane and elastic unloading on the other side. The position of the neutral plane will then be constant. If, as in region I, the strain is a linear function of the distance from the neutral plane, $\varepsilon = -(z-z_2)(\partial\psi/\partial x)$, the strain distribution with two elastic zones must approach the same expression as the first elastic zone disappears. Consequently, (9) gives

$$\tfrac{1}{2}(1-\beta^2)(z_1'-z_2)^2 + 2z_2 = 0$$

and

$$z_2 = -\frac{\{1-\sqrt{[1-(1-\beta^2)z_1']}\}^2}{1-\beta^2} = -\frac{1-\beta}{1+\beta} = -z_1, \quad (12)$$

since $z_1' = 1$.

The first elastic zone disappears when the total strain from regions I and II cancel at $z = 1$. The total strain from region II is not easily found, but with an average value of $z_2 = \overline{z_2}$ the disappearance will occur approximately when

$$-(1-z_1)\left(\frac{\partial\psi}{\partial x}\right)_1 - (1-\overline{z_2})\left\{\frac{\partial\psi}{\partial x} - \left(\frac{\partial\psi}{\partial x}\right)_1\right\} = 0$$

or

$$\frac{\partial\psi}{\partial x} = \frac{z_1-\overline{z_2}}{1-\overline{z_2}}\left(\frac{\partial\psi}{\partial x}\right)_1,$$

where $(\partial\psi/\partial x)_1$ is the maximum curvature.

Approximate values of β may be $\beta \approx 0{\cdot}2$ for aluminium and $\beta \approx 0{\cdot}1$ for steel. The corresponding values of z_1 are $z_1 \approx 0{\cdot}7$ and $z_1 \approx 0{\cdot}8$, while z_2 will range between $-0{\cdot}2$ and $-0{\cdot}7$, and $-0{\cdot}3$ and $-0{\cdot}8$ for the two metals respectively. With average values $\overline{z_2} = -0{\cdot}45$ and $\overline{z_2} = -0{\cdot}55$ the first elastic zone disappears after a curvature reduction in the order of 20 per cent for aluminium and in the order of 15 per cent for steel.

PERTURBATION VELOCITIES AND WAVE VELOCITY

The velocity of a curvature perturbation in the different regions of the plate is equal to the characteristic velocity in the corresponding wave equation. This wave equation is found by performing the integration in (6), or

$$\frac{\partial}{\partial x}\int_{-1}^{1}\sigma z\,\mathrm{d}z - Q = \frac{\partial^2}{\partial t^2}\int_{-1}^{1} uz\,\mathrm{d}z.$$

In region I

$$\frac{\partial}{\partial x}\int_{-1}^{1}\sigma z\,\mathrm{d}z = -\frac{\partial}{\partial x}\left\{\beta^2\int_{-1}^{z_1}(z-z_1)z\frac{\partial\psi}{\partial x}\,\mathrm{d}z + \int_{z_1}^{1}(z-z_1)z\frac{\partial\psi}{\partial x}\,\mathrm{d}z\right\}$$

$$= -\frac{2}{3}\frac{\partial^2\psi}{\partial x^2}\{1-\tfrac{1}{4}(1-\beta^2)(2+3z_1-z_1^3)\},$$

and, since

$$\mathrm{d}u = -(z-z_1)\,\mathrm{d}\psi \quad\text{and}\quad \frac{\partial^2 u}{\partial t^2} = -(z-z_1)\frac{\partial^2\psi}{\partial t^2},$$

$$\frac{\partial^2}{\partial t^2}\int_{-1}^{1} uz\,\mathrm{d}z = -\frac{2}{3}\frac{\partial^2\psi}{\partial t^2}.$$

In region II the moment cannot be determined when the elastic–plastic boundaries shift. However, at the start of region II, and as the first elastic zone disappears,

$$\frac{\partial}{\partial x}\int_{-1}^{1}\sigma z\,\mathrm{d}z = -\frac{\partial}{\partial x}\left\{\int_{-1}^{z_2}(z-z_2)z\frac{\partial\psi}{\partial x}\,\mathrm{d}z + \beta^2\int_{z_2}^{z_1'}(z-z_2)z\frac{\partial\psi}{\partial x}\,\mathrm{d}z + \int_{z_1}^{1}(z-z_2)z\frac{\partial\psi}{\partial x}\,\mathrm{d}z\right\}$$

$$= -\frac{2}{3}\frac{\partial^2\psi}{\partial x^2}\{1-\tfrac{1}{4}(1-\beta^2)(2z_1'^3-3z_1'^2z_2+z_2^3)\}$$

$$+\frac{\partial\psi}{\partial x}(1-\beta^2)(z_1'-z_2)\left\{z_1'\frac{\partial z_1'}{\partial x}-\tfrac{1}{2}(z_1'+z_2)\frac{\partial z_2}{\partial x}\right\},$$

where the last term is introduced by the dependence of z_1' and z_2 on x. By (11) and (12) this term reduces to $-2(z_1'+z_2)(\partial z_2/\partial x)(\partial\psi/\partial x)$ at the start and end of the boundary-shifting phase, with $z_1' = z_1$ and $z_1' = 1$ respectively. After the first elastic zone has disappeared the term is zero. Further, when

$$\mathrm{d}u = -(z-z_2)\,\mathrm{d}\psi, \qquad \frac{\partial^2}{\partial t^2}\int_{-1}^{1} uz\,\mathrm{d}z = -\frac{2}{3}\frac{\partial^2\psi}{\partial t^2}$$

as in region I.

The wave equations can then be written as

$$\frac{\partial^2\psi}{\partial t^2} - \{1-\tfrac{1}{4}(1-\beta^2)(2+3z_1-z_1^3)\}\frac{\partial^2\psi}{\partial x^2} = \tfrac{3}{2}Q \tag{13}$$

in region I, and as

$$\frac{\partial^2\psi}{\partial t^2} - \{1-\tfrac{1}{4}(1-\beta^2)(2z_1'^3-3z_1'^2z_2+z_2^3)\}\frac{\partial^2\psi}{\partial x^2} = \tfrac{3}{2}Q + 3(z_2'+z_2)\frac{\partial z_2}{\partial x}\frac{\partial\psi}{\partial x} \tag{14}$$

in region II, except when the elastic–plastic boundaries shift. In (14) $z_1' = z_1$ at the start of region II and $z_1' = 1$ as the first elastic zone disappears. (13) and (14) show that the equation is basically of the same form in region I and at the head and tail of region II. Surmising that it has the same form even when the elastic–plastic boundaries are shifting, the wave equation for curvature perturbations can be written in characteristic form

$$\frac{\partial^2\psi}{\partial t^2}-c^2\frac{\partial^2\psi}{\partial x^2} = f\left(Q,\frac{\partial\psi}{\partial x}\right)^*. \tag{15}$$

The characteristics of this equation are expressed by

$$\mathrm{d}x = c_\mathrm{I}^+\,\mathrm{d}t = \{1-\tfrac{1}{4}(1-\beta^2)(2+3z_1-z_1^3)\}^{\frac{1}{2}}\,\mathrm{d}t$$

and

$$\mathrm{d}x = c_\mathrm{I}^-\,\mathrm{d}t = -\{1-\tfrac{1}{4}(1-\beta^2)(2+3z_1-z_1^3)\}^{\frac{1}{2}}\,\mathrm{d}t$$

in region I, by

$$\mathrm{d}x = c_\mathrm{II}^+\,\mathrm{d}t = \{1-\tfrac{1}{4}(1-\beta^2)(2z_1^3-3z_1^2z_2+z_2^3)\}^{\frac{1}{2}}\,\mathrm{d}t$$

and

$$\mathrm{d}x = c_\mathrm{II}^-\,\mathrm{d}t = -\{1-\tfrac{1}{4}(1-\beta^2)(2z_1^3-3z_1^2z_2+z_2^3)\}^{\frac{1}{2}}\,\mathrm{d}t$$

at the head of region II, and by

$$\mathrm{d}x = c_\mathrm{II}^+\,\mathrm{d}t = \{1-\tfrac{1}{4}(1-\beta^2)(2-3z_2+z_2^3)\}^{\frac{1}{2}}\,\mathrm{d}t$$

and

$$\mathrm{d}x = c_\mathrm{II}^-\,\mathrm{d}t = -\{1-\tfrac{1}{4}(1-\beta^2)(2-3z_2+z_2^3)\}^{\frac{1}{2}}\,\mathrm{d}t$$

at the tail of region II.

While c_I is constant, c_II will vary with the curvature decrease, going from

$$c_\mathrm{II} = \pm\{1-\tfrac{1}{4}(1-\beta^2)(2z_1^3-3z_1^2z_2+z_2^3)\}^{\frac{1}{2}} \tag{16}$$

to

$$c_\mathrm{II} = \pm\{1-\tfrac{1}{4}(1-\beta^2)(2-3z_2+z_2^3)\}^{\frac{1}{2}} = \pm\{1-\tfrac{1}{4}(1-\beta^2)(2+3z_1-z_1^3)\}^{\frac{1}{2}} = c_\mathrm{I} \tag{17}$$

when the first elastic zone has disappeared. Putting $\partial\psi/\partial x = K$, it is found from (14) that

$$\frac{\partial c_\mathrm{II}^2}{\partial K} = -\tfrac{1}{4}(1-\beta^2)\left\{(6z_1'^2-6z_1'z_2)\frac{\mathrm{d}z_1'}{\mathrm{d}K}-3(z_1'^2-z_2^2)\frac{\mathrm{d}z_2}{\mathrm{d}K}\right\}. \tag{18}$$

* Extending the notation of elastic plate theory

$$Q = \int_{-1}^{1}\tau(z)\,\mathrm{d}z = \int_{-1}^{1}\mu(z)\gamma(z)\,\mathrm{d}z = 2\kappa\mu\gamma(0) = 2\kappa\mu\left(\frac{\partial w}{\partial x}-\psi\right),$$

where μ is elastic rigidity, $\partial w/\partial x$ is relative vertical displacement of the sections, and κ is a factor compensating for the ignored variation in μ and γ over the section. Q does therefore not contain second derivatives of ψ, and can be included in the sourcefunction.

As the curvature decreases z'_1 goes from $z'_1 = z_1 = 1-\beta/1+\beta$ to $z'_1 = 1$ and z_2 goes from $z_2 = -\{[1-\sqrt{(2\beta-\beta^2)}]^2\}/1-\beta^2$ to $z_2 = -(1-\beta/1+\beta) = -z_1$. It is easily shown that

$$\frac{-[1-\sqrt{(2\beta-\beta^2)}]^2}{1-\beta^2} \geq -\frac{1-\beta}{1+\beta}$$

for $0 \leq \beta \leq 1$. Therefore $z'_1 > 0$, $z_z < 0$, $z_1'^2 > z_2^2$, $\partial z'/\partial K < 0$, and $\partial z_2/\partial K > 0$, and the right side of (18) is positive. At both ends of the boundary shifting phase a curvature decrease is thus associated by a decrease in the characteristic velocity. Since the curvature reduction also decreases the width of the elastic zones, it seems an obvious conclusion that the characteristic velocity is a function of the relative thickness of the elastic and plastic zones. The characteristic velocity will then have its maximum value (16) at the start of region II, decreasing monotonically to the value (17) with decreasing curvature.

The state of the material during the first curvature reduction in region II causes the c^+ characteristics originating there to be faster than the c^+ characteristics in region I. Thus, the c^+ characteristics will intersect, causing a breakdown in the continuous solution of the wave equation [11]. This breakdown would be avoided if higher order effects were admitted to the analysis. Alternatively, discontinuities in the dependent variables may be allowed at the intersection. The velocity of the intersection can then be found from the jump conditions.

If $x = x_b$ is a section just in front of the intersection and $x = x_a$ is a section just behind it, the integrated equation of motion (6) can be integrated again in the x-direction,

$$\int_{x_b}^{x_a} \frac{\partial M}{\partial x} \mathrm{d}x - \int_{x_b}^{x_a} Q \, \mathrm{d}x = \int_{x_b}^{x_a} \mathrm{d}x \int_{-1}^{1} \frac{\partial^2 u}{\partial t^2} z \, \mathrm{d}z. \tag{19}$$

If all the integrands are continuous within the limits of integration, this gives the identity $0 = 0$ as $x_b - x_a \to 0$. However, if there are significant discontinuities in the integrands the equation (19) will when $x_b - x_a \to 0$ give a relation between these discontinuities.

The shear force Q is continuous over the bending characteristics, and the second term on the left disappears. The first term on the left in (19) gives

$$\lim_{x_b - x_a \to 0} \int_{x_b}^{x_a} \frac{\partial M}{\partial x} \mathrm{d}x = \lim_{x_b - x_a \to 0} (M_a - M_b) = [M],$$

where $[M]$ denotes the jump in the bending moment from region I to region II.

If U is the velocity of the bending wave, the derivatives $\partial/\partial t$ and $-U(\partial/\partial x)$ of any variable associated with the wave motion are approximately equal,

$$\frac{\partial}{\partial t} \approx -U\frac{\partial}{\partial x}.$$

The term on the right in (19) will therefore give

$$\lim_{x_b - x_a \to 0} \int_{x_b}^{x_a} \mathrm{d}x \int_{-1}^{1} \frac{\partial^2 u}{\partial t^2} z \, \mathrm{d}z = U^2 \left\{ \left(\int_{-1}^{1} \frac{\partial u}{\partial x} z \, \mathrm{d}z \right)_a - \left(\int_{-1}^{1} \frac{\partial u}{\partial x} z \, \mathrm{d}z \right)_b \right\}$$

$$= U^2 \left[\int_{-1}^{1} \frac{\partial u}{\partial x} z \, \mathrm{d}z \right].$$

The jump condition is then

$$[M] = U^2\left[\int_{-1}^{1} \frac{\partial u}{\partial x} z\,dz\right]. \tag{20}$$

The left term is expanded

$$\begin{aligned}
[M] &= \lim_{x_b - x_a \to 0} (M_a - M_b) = \int_{-1}^{1} (\sigma_1 + \sigma_2) z\,dz - \int_{-1}^{1} \sigma_1 z\,dz \\
&= \Delta \int_{-1}^{1} \sigma_2 z\,dz = -\Delta\left(\frac{\partial\psi}{\partial x}\right)\left\{\int_{-1}^{z_2} (z - z_2) z\,dz + \beta^2 \int_{z_2}^{z_1} (z - z_2) z\,dz + \int_{z_1}^{1} (z - z_2) z\,dz\right\} \\
&= -\frac{2}{3}\Delta\left(\frac{\partial\psi}{\partial x}\right)\{1 - \tfrac{1}{4}(1-\beta^2)(2z_1^3 - 3z_1^2 z_2 + z_2^3)\}
\end{aligned} \tag{21}$$

where $\Delta(\partial\psi/\partial x)$ is the first curvature decrement in region II from the maximum reached in region I.

The right term in (19) gives

$$\begin{aligned}
U^2\left[\int_{-1}^{1} \frac{\partial u}{\partial x} z\,dz\right] &= U^2 \lim_{x_b - x_a \to 0} \left\{\left(\int_{-1}^{1} \frac{\partial u}{\partial x} z\,dz\right)_a - \left(\int_{-1}^{1} \frac{\partial u}{\partial x} z\,dz\right)_b\right\} \\
&= -U^2\left\{\frac{\partial\psi}{\partial x}\int_{-1}^{1} (z - z_2) z\,dz - \left(\frac{\partial\psi}{\partial x}\right)_1 \int_{-1}^{1} (z - z_1) z\,dz\right\} \\
&= -\frac{2}{3}U^2\left\{\frac{\partial\psi}{\partial x} - \left(\frac{\partial\psi}{\partial x}\right)_1\right\} = -\frac{2}{3}U^2\Delta\left(\frac{\partial\psi}{\partial x}\right).
\end{aligned} \tag{22}$$

(20), (21) and (22) then give the relation

$$U^2 = 1 - \tfrac{1}{4}(1-\beta^2)(2z_1^3 - 3z_1^2 z_2 + z_2^3), \tag{23}$$

showing that the velocity of the curvature jump is equal to the characteristic velocity at the start of region II as given by (16).

The inhomogeneous terms in the wave equation (15) must cause a dampening of the curvature wave [11], i.e. the curvature in the bend will decrease as the bend propagates along the plate. This curvature decrease will, however, not influence the bend velocity, which only depends on the change in curvature from section to section, and not on the curvature itself.

That the curvature must decrease with time and distance run can also be seen from geometric considerations. The final rotation θ behind the bend is roughly given by

$$\theta = \int \frac{\partial\psi}{\partial x}\,dx, \tag{24}$$

where the integration goes all through the bend. It will be shown shortly that the bend widens. (24) then implies that without curvature decrease the final rotation θ must increase, as shown in Fig. 6. Since its velocity is constant all positions of the wavefront A lies on a straight line through the support. With increasing θ the bend apex B and the rear end of the wave C will on the other hand lie on lines curving downward. As a result the distance AB will increase relative to the distance BC, and in a short while $AB > BC$. However,

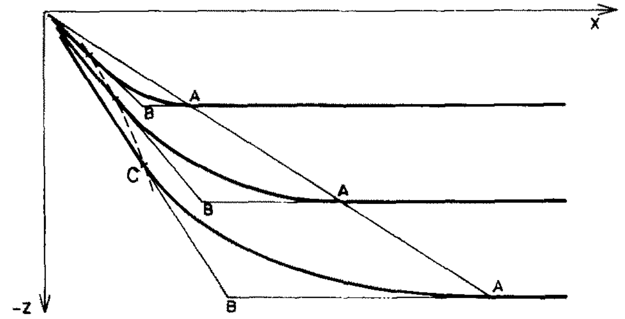

FIG. 6. Bending wave in the plate without curvature dampening. The front of the wave is at A and the rear at C. B is the apex of the tangents.

if tangents are drawn from two points on a curve between which the curvature is monotonically changing without a change of sign, it seems to be a geometric necessity that the distance to the intersection of the tangents is shortest from the point of contact where the curvature is the largest. $AB > BC$ then implies that the curvature is larger at the rear than at the front, in conflict with the previous results. It must therefore be concluded that θ does not increase, and since it seems unlikely that it should decrease, the final deflection angle will be constant. According to (24) the curvature must then decrease as the bend widens. The wave will then appear as shown in Fig. 7, with the corresponding curvature wave shown in Fig. 8.

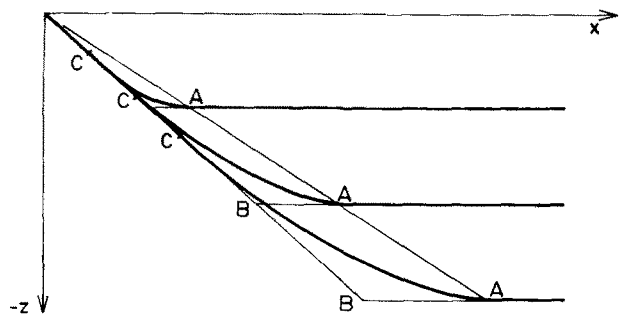

FIG. 7. Bending wave in the plate with curvature dampening.

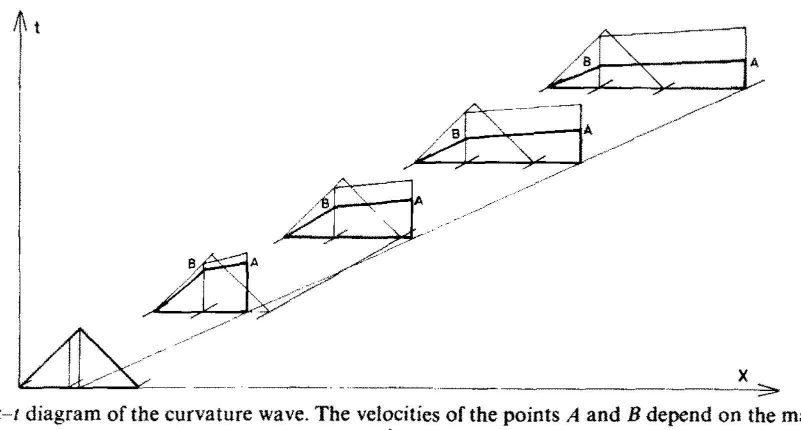

FIG. 8. x–t diagram of the curvature wave. The velocities of the points A and B depend on the material properties only.

CONCLUSIONS

The velocity of curvature perturbations has been found constant, except for the higher values in the first phase of region II. This first curvature decrease will therefore overtake the curvature increase of region I, and leave the rest of the region II behind. A curvature discontinuity will then develop at the front, the curvature jumping from zero to its maximum. While the original waveform may determine the shape of the tail, the front end of the wave therefore develops into the same shape independent of the initial waveform.

According to (23) the front end travels with the velocity

$$U_F = Uc_p = \{1-\tfrac{1}{4}(1-\beta^2)(2z_1^3-3z_1^2z_2+z_2^3)\}^{\frac{1}{2}}c_p \tag{25}$$

while from (17) the rear end travels with the velocity

$$U_R = c_{\text{II}}c_p = \{1-\tfrac{1}{4}(1-\beta^2)(2+3z_1-z_1^3)\}^{\frac{1}{2}}c_p. \tag{26}$$

where

$$z_1 = \frac{1-\beta}{1+\beta}, \qquad z_2 = -\frac{[1-\sqrt{(2\beta-\beta^2)}]^2}{1-\beta^2},$$

$$\beta^2 = \frac{1-\nu_1^2}{1-\nu_2^2}\alpha^2 \quad \text{and} \quad c_p^2 = \frac{E}{\rho_0(1-\nu_1^2)}.$$

When the plate material is elastic-perfectly plastic the workhardening coefficient α^2, and thereby β, is zero, and $z_1 = 1, z_2 = -1$. Both U_F and U_R are then zero, and accordingly no bending wave can travel in such a plate. If the deformations take place under completely elastic conditions $\alpha^2 = 1$. Under these conditions $\nu_2 = \nu_1$ and therefore from (3) $\beta = 1$. Then $z_1 = z_2 = 0$ and $U_F = U_R = c_p$, i.e. the perturbation velocity in bending does not change between regions I and II, and no curvature discontinuities will develop. For all other values of α^2, $U_F > U_R$, and the wave will change shape and spread out as it travels along the plate. In Fig. 9 U_F and U_R are plotted as proportions of the elastic plate velocity c_p against the value of β^2, together with the corresponding value of the plastic plate velocity c_{pp}.

As seen from the slopes of the curves at small values of β^2, and thereby α^2, the results are significantly different from the results in the perfectly plastic case even for very small values of the coefficient of workhardening α^2. Indeed, for $\beta^2 = 0{\cdot}01$, corresponding to $\alpha^2 \approx 0{\cdot}0085$, the velocity U_F is already 75 per cent of the elastic plate velocity.

The fact that the bending wave runs faster than the plastic longitudinal wave is thus a necessary consequence of the strain reversals and the accompanying elastic deformations that occur in the bend, and must not be confused with strain rate effects.

In Fig. 8 the velocities of points A and B are given by (25) and (26) respectively. It is seen from Fig. 9 that the horizontal distance between the points increases with time and distance run. At the same time the vertical distance between the points will not increase. Therefore, although the characteristic velocity is undetermined between A and B, and the actual shape of the curve between the points consequently is unknown, the wave will develop a flat top. A section at a distance from the supports will thus as the wave appears experience a moment change given by a stepfunction.

As already noted, the velocity of the bending wave lies between the elastic and the plastic longitudinal wave velocities, and the prestress level it experiences is the yield stress of the plate material. The stress will then be the same in front of and behind the bend. The conclusions of the present paper are therefore also valid for the impulsively loaded plate.

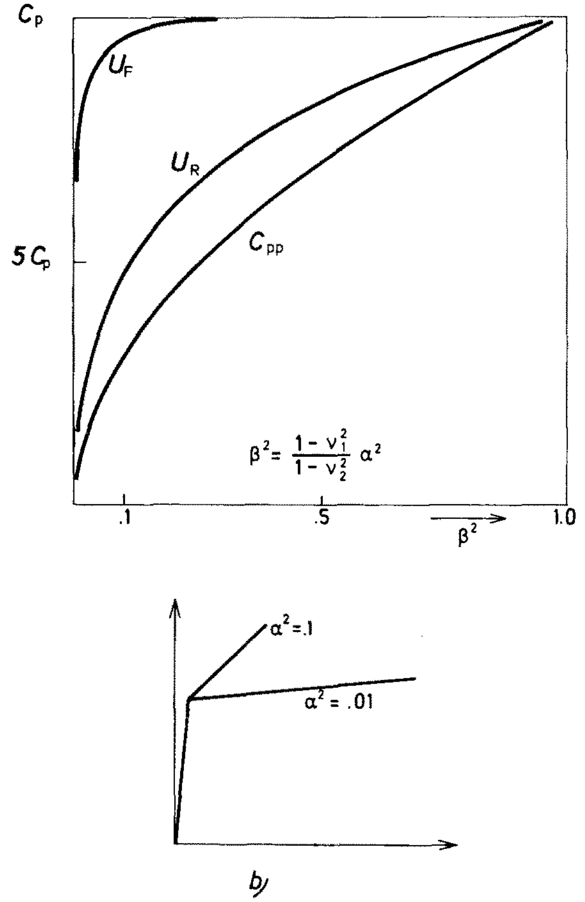

FIG. 9. (a) The elastic and plastic plate velocities c_p and c_{pp}, and the velocities U_F and U_R of the front and rear end of the bending wave, for different values of the parameter β^2. (b) Stress–strain curves with workhardening coefficients $\alpha^2 = 0{\cdot}01$ and $\alpha^2 = 0{\cdot}1$. For small values of α^2 $\beta^2 \approx \frac{4}{3}(1-\nu_1^2)\alpha^2$.

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Абстракт—Работа обсуждает изгобные волны в двухлинейных упругопластических пластинках, предпрятых выше точки текучести. Находится, что изменение направления деформаций, связаное сдвижением изгиба, придает ему скорость, которая зависит только от свойств материала пластинки и располагается между скоростями продольной, упругой и пластической волны. Констатируется, что отрезок максимальной кривизны обладает большей скоростью чем другие отрезки при изгибе. Таким образом волна будет вызывать разрыв кривизны на фронте, где скачуок ее колеблется от нуля до максимума. Медленнее движущийся конец волны остается позади и таким образом изгиб распространяется.